Review/Practice:

Find ① $\int_0^{\pi} \sin(x) e^{\cos(x)}\,dx$

② $\int_0^1 \sqrt{1+\sqrt{x}}\,dx$

These are integrals of differential forms

e.g. $\sin(x) e^{\cos(x)}\,dx$ is a differential 1-form.

The notation allows us to change variables nicely so the chain rule is taken into account.

Operations with functions & differential forms:

$$d(x^2+5x) = (2x+5)\,dx$$

$$\Delta(x^2+5x) \approx (2x+5)\Delta x$$

if x changes from 5 to 5.01

$\Rightarrow$ $(5.01)^2+5(5.01)-(5^2+5\cdot5) \approx (2(5)+5)(.01)$

change in function $\approx$ ____ $\begin{pmatrix}\text{change}\\ \text{in}\\ x\end{pmatrix}$

our example

$$0.1501 \approx 15(.01)$$
$$= .15$$

Several variables.

$$d\left(x^2y + 5xy^3\right)$$
$$= \left(2xy + 5y^3\right)dx + \left(x^2 + 15xy^2\right)dy$$

$$\Delta\left(x^2y + 5xy^3\right) \approx \left(2xy + 5y^3\right)\Delta x + \left(x^2 + 15xy^2\right)\Delta y$$

$(1) \to 1.01$ in x

$(-1) \to -1.02$ in y

$$(1.01)^2(-1.02) + 5(1.01)(-1.02)^3 - \left(1^2(-1) + 5(1)(-1)^3\right)$$

$$\approx \left(2(1)(-1) + 5(-1)^3\right).01 + \left(1^2 + 15\cdot 1\cdot(-1)^2\right)(-.02)$$

should be close.

Find ① $\displaystyle\int_{x=0}^{\pi} \sin(x)\, e^{\cos(x)}\, dx$

② $\displaystyle\int_0^1 \sqrt{1 + \sqrt{x}}\; dx$

(1) $u = \cos(x)$ $\qquad du = \underline{-\sin(x)\,dx}$

$u = \cos(\pi) = -1$ $\qquad -du = \sin(x)\,dx$

$u = \cos(0) = 1$

$$\Rightarrow \int_{u=1}^{-1} e^{u}\,(-du) = -\int_{1}^{-1} e^{u}\,du$$

$$= +\int_{-1}^{1} e^{u}\,du \underset{\text{(FTC)}}{=} e^{u}\Big|_{-1}^{1} = \boxed{e^{1} - e^{-1}}.$$

(2) Let $\underline{u = 1 + \sqrt{x}}$ $\Rightarrow du = \frac{1}{2}x^{-1/2}\,dx$

$= \frac{1}{2\sqrt{x}}\,dx$

Not there

$\Downarrow$

$u - 1 = \sqrt{x}$

$(u-1)^2 = x$

$dx = 2(u-1)\,du$

Bounds

$u = 1 + \sqrt{0} = 1$

$u = 1 + \sqrt{1} = 2$

$$\int_{x=0}^{1} \sqrt{1+\sqrt{x}}\,dx = \int_{u=1}^{2} (\sqrt{u})(2(u-1)\,du)$$

$$= \int_{u=1}^{2} \left(2u^{3/2} - 2u^{1/2}\right)du = \frac{2\cdot 2}{5}u^{5/2} - \frac{2\cdot 2}{3}u^{3/2}\Big|_{1}^{2}$$

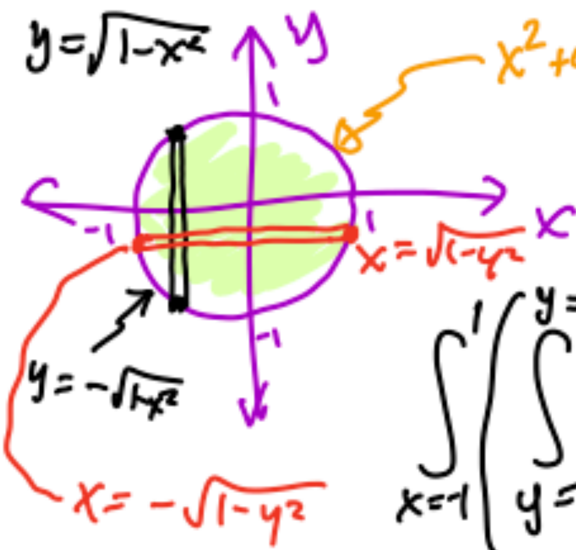
$$= \frac{4}{5}\cdot 2^{5/2} - \frac{4}{3}\cdot 2^{3/2} - \left(\frac{4}{5}\cdot 1 - \frac{4}{3}\cdot 1\right)$$

$$= \boxed{\frac{4}{5}2^{5/2} - \frac{4}{3}2^{3/2} + \frac{8}{15}} \qquad \frac{12}{15} - \frac{20}{15}$$

More multivariable integrals.

Example. Find $\iint (xy-3x)\,dA$ over the region that is inside the unit circle.

$dA = dx\,dy$ or $dy\,dx$ (little piece of area)



(A) Vertical Strips first.

$$\int_{x=-1}^{1}\left(\int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} (xy-3x)\,dy\right)dx$$

$$= \int_{x=-1}^{1} \left(\frac{xy^2}{2} - 3xy\right)\Bigg|_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= \int_{x=-1}^{1} \left[\frac{x(1-x^2)}{2} - 3x\sqrt{1-x^2}\right] - \left[\frac{x}{2}(1-x^2) + 3x\sqrt{1-x^2}\right] dx$$

$$= \int_{x=-1}^{1} -6x\sqrt{1-x^2}\, dx$$

$$u = 1-x^2 \Rightarrow du = -2x\, dx$$

$$-\tfrac{1}{2} du = x\, dx$$

Bounds: $u = (1-(-1)^2) = 0$

top $u = (1-1^2) = 0$

$$= \int_{u=0}^{0} -6\sqrt{u}\,\left(-\tfrac{1}{2} du\right) = 3\int_0^0 \sqrt{u}\, du = \boxed{0}$$

$2u^{3/2}$

(B) Horizontal strips first.

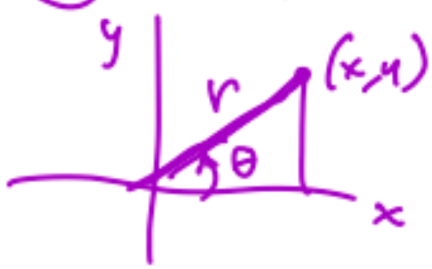
$$\int_{y=-1}^{1} \left(\int_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} (xy - 3x)\, dx \right) dy$$

$$= \int_{y=-1}^{1} \cancel{\frac{yx^2}{2} - \frac{3x^2}{2}}\ (y-3)\frac{x^2}{2} \Bigg|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dy$$

$$= \int_{y=-1}^{1} \frac{(y-3)(1-y^2)}{2} - \frac{(y-3)(1-y^2)}{2} \, dy$$

$$= \int 0 = \boxed{0}.$$

(C) Change to polar coordinates!



$x = r\cos\theta$
$y = r\sin\theta$

$$dA = dx\,dy = dx \wedge dy$$

wedge product of differential forms.

$$dx = (\cos\theta)\,dr - r\sin\theta\,d\theta$$
$$dy = \sin\theta\,dr + r\cos\theta\,d\theta$$